SOME REMARK ON THE MINIMUM MODULUS AND THE DISTRIBUTION OF ZEROS

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1. Introdution

When a function f is holomorphic in the unit disk D, it is known that there exist some relations between the distribution of its zeros and its minimum modulus on circles. For example, in [1], Bonar and Caroll proved a theorem that if the zeros of f lie on p equi-spaced radii of D, then for any sequence of circles $J_n : |z| = r_n$ the minium moduls of f on J_n must be bounded. To prove this theorem they used the well known formulas ([2], p. 437) relating the Fourier coefficients of log $|f(re^{i\theta})|$ to the distribution of zeros of f, and probably, this seems to be the main reason why they happened to consider the minimum modulus of f on circles centered at the origin. But in rder to study the behaviour of f from the standpoint of the minimum modulus, it seems to be better to consider the minimum on arbitrary closed Jordan curves, because we have an example f such that $C(f) < J(f)$. Here $C(f)$ or $J(f)$ denotes respeertively sup min $\{|f(z)| : |z| = r\}$ or sup min $\{|f(z)| : z \in J\}$ where the sup is taken over all circles : $|z| = r$ $(0 < r < 1)$ or all closed Jordan curves J in D surrounding the origin. Thus we have become concerned with the following question. What conditions does f have to satisfy in order that $J(f)$ may be finite ? The aim of this remark is to give a partial answer to this questioin.

2. A fundamental lemma

We denote by S_0 the set of closed Jordan curves in the unit desk D, surrounding the origin and S'_0 the set of curves belonging to S_0 which are symmetric with respect to the real axis. And further, let H be the family of functions holomorphic in D. For an element J of S_0 , we denote by ϕ with $\phi(0) = 0$ the conformal mapping from the interior of J, say D_J, onto D and consider the Green's function for D_J

$$g(z) = -\log|\phi(z)|$$

with pole at $z = 0$. Then, if h is a real-valued function continuous on an analytic curve J, its Fourier coeffcients will be defind by

$$c_j = -\frac{1}{2\pi}\int_J h(z)\phi(z)^{-j} * dg(z) \quad (j = \pm 1, \pm 2 ...)$$

where J is to be traversed in the positive sense, $*dg(z)$ denotes $\frac{\partial g}{\partial n}$ ds by definition and $\frac{\partial g}{\partial n}$ means the outer normal. Then, we have

Proposition. $c_0 - |c_j| \geq \min_{z \in J} h(z)$ for any $j \neq 0$.

Proof. Consider the Fourier coefficients of a non-negative funcution

$$h(z) - \min_{z \in J} h(z)$$

and note

$$\int_J \phi(z)^{-j} * dg(z) = 0 \text{ for any } j \neq 0.$$

Then a standard argument implies Proposition.

We shall apply this proposition to the case in question. So, let f be a function in H with $f(0) \neq 0$ and J an analytic curve in S_0, which does not pass through any of the zeros of f. For this J, we consider again the conformal mapping ϕ and the Green's function g as in the paragraph preceding Proposition and denote by

$$c_j=-\frac{1}{2\pi}\int_J \{\log|f(z)|\}\,\phi(z)^{-j} * dg(z)$$

the Fourier coefficients of $\log|f(z)|$ on J. Using almost the same methods as in [2], we shall derive the standard formulas below involving c_j and the distribution of zeros of f. Namely, we shall give

Lemma. *Let* $z_1, z_2, \ldots, z_n$ *be the zeros of* f *in* D_J, *listed according to multiplicity and denote by* A_m, a_m *the* m*-th Maclaurin cofficients of* $\{\frac{z}{\zeta(z)}\}^j$, $\log f(z)$ *respectively. Then*

(1) $$c_0=\log|f(0)|-\sum_{k=1}^{n}\log|\phi(z_k)|$$

(2) $$c_j=\frac{1}{2j}\sum_{m=1}^{j} m\, a_m A_{j-m}+\frac{1}{2j}\sum_{k=1}^{n}\{\phi(z^{-j}{}_k)-\overline{\phi(z_k)^j}\}\qquad (j>0)$$

where as for logarithm some determination is to be used.

Proof. The first equality (1) easily follows from applying Green's formulas to $\log|f(z)|$ and $g(z)$ which are harmonic in the domain

$$D_J-\{z:|f(z)|\le\varepsilon\}-\{z:|\phi(z)|\le\varepsilon\}$$

with some small $\varepsilon>0$. To show (2), we first set

(3) $$c_j'=\int_J \{\log f(z)\}\,\mathrm{Re}\{\phi(z)^{-j}\} * dg(z)$$

(4) $$c_j''=\int_J \{\log f(z)\}\,\mathrm{Im}\{\phi(z)^{-j}\} * dg(z)$$

Then obviously there holds

(5) $$-2\pi c_j=\mathrm{Re}c_j'+i\ \mathrm{Re}c_j''.$$

Next, we represent J by the equation $z=z(s)$ $0\le s\le L$ where L is the length of J, and evaluate the right-hand sides of (3), (4) by integration by parts, rewriting them in terms of real integrals. And again we represent the results in terms of complex integrals. Then (3), (4) become as follows:

(3)′ $$c_j' = \frac{1}{2ji}\int_J \frac{f'(z)}{f(z)}\{\phi(z)^j - \phi(z)^{-j}\}\,dz$$

(4)′ $$c_j'' = \frac{1}{2j}\int_J \frac{f'(z)}{f(z)}\{\phi(z)^j - \phi(z)^{-j}\}\,dz$$

Finally, the residue theorem yields (2) together with (5).

3. Estimation of the minimum modulus

Now our principal result is the following

Theorem 1. *Let* f *be a functon in* H *and* p *the multiplicity of the origin as a possible zero of* f. *Further, denote by* a_j *the* j-*th Maclaurin coeffcient of* $\log\frac{f(z)}{z^p}$ *where some determination of logarithm may be used. If the zeros of* f *lie on the radius* (0, 1), *then there holds*

(6) $$\min\{|f(z)| : z \in J\} \leq \exp\{\mathrm{Re}\ a_0 + |a_1|\}$$

for any J *in* S'_0.

Proof. Without loss of generality we may assume that J is analytic. Further, if f vanishes on J, there is nothing to be proved. So let $f(z) \neq 0$ on J and consider, to make use of Lemma, a holomorphic function F(z) such that

(7) $$f(z) = z^p F(z)\exp\{a_0 + a_1 z\}$$

Then clearly $F(0) = 1$, while racalling that a_0, a_1 are given by the Maclaurin expansion

$$\log\frac{f(z)}{z^p} = a_0 + a_1 z + a_2 z^2 + \cdots$$

we obtain

$$\log F(z) = a_2 z^2 + a_3 z^3 + \cdots$$

Therefore, letting $z_1, z_2, \ldots, z_n$ be the zeros of F in D_J, we apply Lemma with j= 1 to F. Then by the proposition, we get

$$-\sum_{k=1}^{n}\log|\phi(z^k)| - \frac{1}{2}\left|\sum_{k=1}^{n}\{\phi(z^k)^{-1} - \overline{\phi(z^k)}\}\right| \geq \min\{\log|F(z)| : z \in J\}$$

Since, J is symmetric with respect to the real axis and z_k is positive, $\phi(z_k)$ also becomes positive, if we let $\phi'(0) > 0$. Hence, making note of an elementary inequality

$$x - x^{-1} > 2 \log x \quad (x > 1)$$

and then observing (7), we conclude that (6) is valid.

In the next theorem, we shall put on the function f the assumptions such that its zeros lie on the radius (0, 1) and $f(z) = \overline{f(\bar{z})}$, and try to estimate from above not $C(f)$ but $J(f)$. Therefore, in order that this theorem may not become vain, we have to find an f which additionally satisfies $C(f) < J(f)$. For this aim, let A be the only poditive root of the equation

$$x^4 + 2x^3 - 1 = 0$$

and set $a = (1-A)(1+A)^{-1}$. Then the function

$$f(z) = \frac{a-z}{1-az} \exp\left(-\frac{1-z}{1+z}\right)$$

becomes such one. To see this, we have only to consider its minimum modulus on circles : $|(a-z)(1-az)^{-1}| = t \ (a<t<1)$. Now we shall give

Theorem 2. *Let* f *be a function in* H *and suspose that its zeros lie on the radius* (0, 1). *If its Maclaurin cofficients are all real,* i. e. , $f(z) = \overline{f(\bar{z})}$, *then we have*

$$J(f) \leq \exp\{\mathrm{Re}\ a_0 + |a_1|\}.$$

Proof. Let J be any element of S_0 and consider the component of the open set $D_J - (-1, 1)$ whose boundary contains the origin and which is included in the upper half of D. Then its boundary contains a Jordan arc J_0 which is a subarc of J and which connects a point of (0, 1) with one of (−1, 0) in the half disk. So we denote by $\overline{J_0}$ the reflection of J_0 with respect to the diameter (−1, 1) and apply Theorem 1 to the element $J_0 \cup \overline{J_0}$ of S'_0. Thus, our assertion follows.

Remark. If there is a sequence of J_n in S_0 such that $\min\{\,|f(z)|:z\in J_n\}\to\infty$ as $n\to\infty$, f is called annular [1]. Therefore, we see that such an f described in Theorem 2 is not annular.

References

[1] D. D. Bonar and F. W. Carroll, *Distribution of* a-*points for unbounded analytic functions*, J. reine angew. Math. 253(1972), 141-145.

[2] L. A. Rubel, *A Fourier series method for entire functions*, Duke Math. J. 30(1963), 437-442.