岐阜薬科大学基礎教育系紀要、第８号、1996年

SOME CONDITION FOR HARMONICITY

AKIO OSADA

1.Introduction

Let K be a non-empty compact set on the complex plane and m a finite positive Borel measure on K. Given an analytic function f defined in an open segment (0, a), where the positive number a , including the possible case of the infinity a, should be chosen in order to satisfy $|z-Z|<a$ for any Z in K and any z in a complementary component of K, our problem is to characterise f so that the function

$$\int_K f(|z-Z|)dm(Z)$$

could be harmonic in the complementary component of K in question. Now in this paper, a partial answer for this problem is given where the compact set K and the measure m are restricted to the unit circle C and Lebesgue measure on C respectively. Our result, however, improves the former theorem in [1].

2. The fundamental lemma

For an analytic function F(x) in the open segment (0, 4), the n-the Taylor coefficient of f at the point $x=1+r^2$ is denoted by a (n, r). Then we shall prove the following

Lemma. Let F be given as above and put

$$\{[x-(1-r)^2]\ \ [(1+r)^2-x]\}^{-1/2} = g(x, r)$$

where $0<r<1$ and $(1-r)^2<x<(1+r)^2$. Moreover assume that a $(2m-1, r) = 0$ for any r $(0<r<1)$ and each m. Then, if

$$\int F(x)g(x, r)dx=0$$

for any r $(0<r<1)$ where the integral is taken over the open segment $((1-r)^2, (1+r)^2)$, F vanishes constantly.

Proof. First, we expand F at $x=1+r^2(-1<r<1)$.
And so we have

$$F(x)=\sum_{n=1}^{\infty} a(n, r)\{x-(1+r^2)\}^n.$$

Changing the variable x to another variable t by the relation

$$x=1+r^2+2rt,$$

then, the above integral

(1) $$\int F(x)g(x, r)dx$$

becomes

$$\sum_{n=0}^{\infty} a(n, r)\ (2r)^n \quad \int_{-1}^{1} t^n(1-t^2)^{-1/2}\,dt,$$

which vanishes for any r $(0<r<1)$ by the assumption. Here, it follows from elementary calculations that

$$\int_{-1}^{1} t^n(1-t^2)^{-1/2}\,dt$$

vanishes when n is odd and takes the value A(n) when n is even where A(n) is

$$\pi\,(n-1)\ (n-3)\ldots 3/\{n\ (n-2)\ \ldots 2\}.$$

Therefore, we see that the Taylor coefficients a (n, r) of F at $x=1+r^2$ satisfy the codition that

(2) $$\sum_{m=0}^{\infty} a(2m, r)(2r)^{2m} \quad A(2m)=0 \text{ for any } r(-1<r<1)$$

and particuiarly the odd coefficients a (2m-1, r) are not restricted in any-way.

Our remaining task is to show that (2) implies $a(m, r)=0$ for every m and any r ($-1<r<1$).

For this end, we first expand F at two different points $x=r$ and $x=R$ ($-1<r, R<1$). Namely we have the relations

$$(3) \qquad F(x)=\sum_{n=0}^{\infty} a(n, r)\{x-(1+r^2)\}^n = \sum_{j=0}^{\infty} a(j, r)\{x-(1+R^2)\}^j$$

for x in $I(r) \cap I(R)$ where $I(r)$ denotes the convergence interval of the corresponding Taylor series at $x=1+r^2$. Then, as is easily shown, it holds

$$F^{(n)}(x)=\sum_{j=n}^{\infty} a(j, r)\, j\,(j-1)\dots(j-n+1)\,\{x-(1+R^2)\}^{j-n}.$$

Hence, using

$$a(n, r)=F^{(n)}(1+r^2)/(n!),$$

and

$$a(n, R)=F^{(n)}(1+R^2)/(n!),$$

we obtain

$$(4) \qquad a(n, r)=1/(n!)\sum_{j=n}^{\infty} a(j, r)\, {}_jP_n\, \{r^2-R^2\}^{j-n}$$

and

$$(4)' \qquad a(n, R)=1/(n!)\sum_{j=n}^{\infty} a(j, R)\, {}_jP_n\, \{R^2-r^2\}.$$

These relations play important roles for our goal. In fact, from (4) and (4)$'$ together with our assumption, we have

$$a(2m, r)-a(2m, R) = -1/(2m!)\sum_{k=m}^{\infty} {}_{2k}P_{2m}(r^2-R^2)^{2(k-m)}\{a(2k, r)-a(2k, R)\}.$$

Therefore,

$$\{a(2m, r)-a(2m, R)\}/(r-R) = -1/(2m!)\{(2m!)\,[a(2m, r)-a(2m, R)]/(r-R)+$$

$$\sum_{k>m} P(r^2-R^2)^{2(k-m)}/(r-R)\{a(2k, r)-a(2k, R)\}.$$

Letting $r \to R$, we get

$$a'(2m, R) = -a'(2m, R)$$

and hence conclude that $a(2m, r)$ should be a constant function of $r(-1<r<1)$ for every m.

Thus, we can set

$$a(2m, r) = a(2m),$$

which will be used to attain our purpose in connection with (2). In fact, making use of (2), we have

$$\sum_{m=0}^{\infty} a(2m)A(2m)\, 4^m r^{2m} = 0$$

for any $r(-1<r<1)$. Consequently, for each m,

$$a(2m) = 0,$$

which completes the proof.

3. Now that our fundamental lemma has been proved, we shall be in the situation to show the following

Theorem. Let f be analytic in the segment (0, 4) with $f(1)=0$.Moreover assume that each of the odd Taylor coefficients of the function $f'(x)+x f''(x)$ vanishes at the point $x=1+r^2$ for any r $(-1<r<1)$. Then, a necessary and sufficient condition that

$$\int_0^{2\pi} f(|z-Z|)dt$$

may be harmonic in the unit disk $|z|<1$ is that

$$f(x) = \log(x)$$

where $Z=\exp(it)$.

Proof. Set

$$\int_0^{2\pi} f(|z-Z|)dt=u(f, z).$$

Then, as in [1] ,

$$\Delta u(f, z)=0$$

if and only if $\int\{f'(x)+xf''(x)\}g(x, r)dx=0$ for any $r(0<r<1)$. Therefore, it follows from Lemma that $f'(x)+xf''(x)$ vanishes constantly.

And so noting $f(1)=0$, we conclude

$$f(x)=\log(x).$$

In this process, we should emphasize that the infinite series given by the left hand side of (2) possesses a positive radius of convergence.

First of all, put

$$4^m a(2m)A(2m)=b(m).$$

Concerning about the factor A(2m), we see

$$\begin{aligned}&\log A(2m)\\&<\log\pi+\int_1^{m+1}\log\{1-1/(2x)\}dx\\&=m\log\{1-1/[2(m+1)]+1/2\log(m+1/2)\\&\quad-\log(m+1)+0(1),\end{aligned}$$

which implies

$$\overline{\lim_{m\to\infty}}\, 1/m\log A(2m)\leqq 0.$$

On the other hand, we get, a priori,

$$\overline{\lim_{m\to\infty}}\, 1/m\,|a(2m)|<0.$$

These two inequalities give the desired conclusion.

REFERENCES

[1] A.Osada. The Center of Jordan Domains,
Bull. Lib. Arts Gifu Pharm. Univ., (1995)