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A COUNTER EXAMPLE TO HARMONICITY

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1. Introduction

In the former papers [1], [2], we handled the problem of characterizing the kernel G by which the induced integral could be harmonic in the unit disk $|z|<1$. We conjectured in the course of the process that the kernel must be logarithmic. Our present result, however, shows that the non-logarithmic kernels could be able to make the corresponding integrals harmonic in the unit disk $|z|<1$.

2. Let f be an analytic function in the segment $(0,4)$ on the real axis satisfying $f(1)=0$ and consider the function in the unit disk $|z|<1$,

$$u(z)=\int f(|z-Z|^2)dt$$

where $Z=\exp(it)$ and the integral is taken over the interval $[0,2\pi]$. Then the harmonicity of $u(z)$ is equivalent to the validity of the equality

(2) $$\int \tilde{f}(x)g(x,r)dx=0 \quad \text{for any } r\ (0<r<1)$$

where the integral is taken from $(1-r)^2$ to $(1+r)^2$ on the real axis,

$$\tilde{f}(x)=f''(x)x+f'(x)$$

and

$$g(x,r)=\{x-(1-r)^2\}^{-\frac{1}{2}}\ \{(1+r)^2-x\}.$$

Consequently our concern amounts to the characterization of $F(x)$ such that

(2)' $$\int F(x)g(x,r)dx=0 \text{ for any } r\ (0<r<1)$$

where the integration interval is the same one as in (2). Our first conjecture is that the function $F(x)$ satisfying (2)' would be zero through out on that interval,namely,that in terms of $f(x)$, it would coincide with

$$c\log x$$

where c is a constant. In fact, we see that a simple differential equation

$$f''(x)x+f'(x)=0$$

gives $c\log x$ as its solution with $f(1)=0$.

3. This paper, however, presents a very simple counter example to the above conjecture. First of all, using the relation

$$x=1+r^2+2rt$$

we change the variable x to another one t. Then (2)' is rewritten as follows;

(2)" $$\int_{-1}^{1} F(1+r^2+2rt)(1-t)^{-\frac{1}{2}}\,dt=0 \quad \text{for any } r\ (0<r<1)$$

We take the function $\log x$ for $F(x)$. Then using

$$\log(1+r^2+2rt)=\log(2r)+\log\{t+a(r)\}$$

where $a(r)=(1+r^2)/(2r)$, the left hand side of (2)" becomes

(3) $$\pi\log(2r)+\int_{-1}^{1}(1-t)^{-\frac{1}{2}}\log(t+a(r))dt.$$

In the last integral, changing the variable t by the relation $t=\sin(z)$, we see that the second term of (2) is equivalent to

(3)' $$\int\log(\sin(z)+a(r))dz$$

where the integral is taken on the segment $[-\pi/2, \pi/2]$. In order to find the value of the integral (3)', which is noted by $I(a)$, where $a=a(r)$ and r is fixed. Then we differentiate $I(a)$. Using the well known relation

$\tan(z/2)=x$, we get

$$I'(a)=2\int_{-1}^{1}\{a(1+x^2)+2x\}^{-1}\,dx$$
$$=\pi(a^2-1)^{-\frac{1}{2}}$$

Consequently, by integrating $I'(a)$, we get

$$I(a)=\pi\log\{a+(a^2-1)^{\frac{1}{2}}\}+I(1). \tag{4}$$

On the other hand, elementary calculations imply that

$$I(1)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\log\{1+\sin(z)\}\,dz$$
$$=\int_{-1}^{1}\log\{1+\cos(z)\}\,dz$$
$$=\pi\log 2+4\int_{0}^{\frac{\pi}{2}}\log\{\cos(z)\}\,dz$$
$$=-\pi\log 2.$$

Hence, by virtue of (4), we find that the equality

$$I(a)=\pi\log\{[(a^2-1)^{\frac{1}{2}}+a]/2\}$$

holds. Noting $a=(1+r^2)/(2r)$, it is easy to show that

$$I(a)=-\pi\log(2r).$$

Therefore we conclude that the equality (2)' is valid if we take $F(x)=\log x$.

4. Relating the above conjecture, we consider some simple problem as follows;

Let $G(x)$ be an analytic function in the interval $[1, \infty]$ on the real axis and put

$$H(x,r)=\{(x-1)(r-x)\}^{-\frac{1}{2}}$$

Then we show the following

Remark. If the integral

$$\int_{1}^{r}G(x)H(x,r)\,dx$$

vanishes for any r $(1<r)$ and $G(x)$ is monotone on the interval $[1, \infty]$ in question, then $G(x)$ vanishes everywhere in $[1, \infty]$. To see this, we first

note that

$$H(x,r)dx = \sin^{-1}\{(2x-r-1)/(r-1)\}.$$

Next making use of the integration by parts, we get that

(5) $\int_1^r G'(x)\{\sin^{-1}[(2x-r-1)/(r-1)]\}'dx = 0$ for any r $(1<r)$

where the differentiation is taken with respect to r. Here, as is easily shown, we have that

$$[\sin^{-1}\{(2x-r-1)/(r-1)\}]' = -(r-1)^{\frac{1}{2}}\{(r-1)^2(r-t)\}^{-\frac{1}{2}}$$

Consequently, by virtue of (5), we see that

$$\int_1^r G'(x)(r-1)^{\frac{1}{2}}(r-t)^{-\frac{1}{2}}dx = 0 \quad \text{for any } r\ (1<r).$$

Using the monotonity of G(x), we conclude that

$$G(x) = 0$$

which is desired.

REFERENCES

[1] A. Osada. The center of Jordan domains, Bull. Lib. Arts Gifu Pharm. Univ., (1995)

[2] A. Osada. Some condition for harmonicity, Bull. Lib. Arts Gifu Pharm. Univ.,(1996)