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Some Problems of Annular Functions

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1. We take the domains of two types, namely, the disk which has the hyperbolic boundary and the plane which has the parabolic boundary, and consider how the regular functions would behave near the boundary. Exactly speaking, our problems is to find out how the distribution of a points of a -regular function f would be restricted if the minimum of $|f|$ on a sequence of closed Jordan curves, expand－ing to the boundary, increases to the infinity.

2. Definition

According to the well known theorem of Wiman, if an entire function f, which is not constant, has an order $a(f)$ less than $1/2$, then there exists a sequence $\{r_n\}$ such that $m(r_n)\to\infty$. Here $m(r)=\min\{|f(z)|; |z|=r\}$, $a(f)=\overline{\lim_{r\to\infty}}[\log\log M(r)]/\log r$ and $M(r)=\max\{|f(z)|; |z|=r\}$. Therefore, if we take an arbitrary sequence z_n such that the convergence index of $|z_n|$, which is equal to the order of f, satisfies the following inequality

(1) $\quad a(f)=\inf\{k; \sum_n |z_n|^{-k}<\infty\}<1/2.$

And define an entire function by the infinite product

$$\prod_n (1-z/z_n).$$

Then we find that $m(r_n)\to\infty$

where $r_n=|z_n|$. Since the condition (1) is independent of z_n, we can conclude that the plane has the week boundary in the following meaning : the distribution of the zeros of some kinds of entire functions is not restricted in spite of the existence of $\{r_n\}$ such that

$$m(r_n) \to \infty.$$

3. Hyperbolic case

In the case of the disk $|z|<1$, different from the case of the plane, the boundary $|z|=1$ may be so strong that the distribution of the zeros of regular functions must be restricted if the minimum condition satisfies

$$m(r_n) \to \infty.$$

For this end, Bagemihl-Erdös considered some special kinds of regular functions. Namely, a regular function is called annular if and only if there exists a sequence J_n of closed Jordan curves such that

$$\min\{\,|f(z)|\,;\ z \in J_n\} \to \infty$$

and $\{J_n\}$ uniformly converges to the boundary $|z|=1$. If we can choose a sequence $\{\,|z|=r_n\}$ of circles in place of $\{Jn\}$, the function is called strongly annular. For a regular function f, we consider the zero set of f, i.e.,

$$Z(f) = \{z\ ;\ f(z)=0\}$$

and define the set of all cluster points of $Z(f)$ by $Z'(f)$. Hence, $Z'(f)$ can not be empty if the zero set $Z(f)$ is an infinite set. This is an important observation because we can not expect the same result in the complex plane.

4. Constructions of annular functions

We can give some strongly annular functions by means of gap series (Lusin-Privarov, 1925)

(2) $\quad \sum_k a_k z^{n_k}$ [ak]

and infinite products (Wolff, 1928)

(3) $\prod_{k}(1-a_k z^{n_k})$.

These cases, however, can not satisfy our desire. In fact, it is almost impossible that we find out the distribution of the zero in the case of (2), and it is regretful in the case of (3) that the zero set of the function uniformly converges to the C : $|z|=1$, namely to the boundary of the domain, which gives

$$Z'(f)=C.$$

Consequently, it is so natural to ask the following questions because we can not find out some other methods for constructing annular functions and receiving necessary information about the distribution of their zeros ;

(Q) : If f is annular, is it valid $Z'(f)=C$?

(Q) : Does there exist an annular function which has Fatou points ?

5.

To settle the questions Q_1, Q_2, we use the approximation theorems by Mergelyan[1] and Arakelian[2]. For the question Q_1, Barth-Schneider[3] succeeded in the construction of an example of an annular function such that

$$Z'(f)=\{1\}.$$

But, regretfully, it is not clear whether this example is strongly annular or not.

Therefore, we prove the following

THOREM. Let $C_j=\{z\,;\,z=z_j(t),\ 0\leq t\leq 1\}$ $(j=1,2)$ be two Jordan arcs such that

(4) $z_1(0)=iy$ $(0<y_1<1)$ and $z_2(0)=iy$ $(-1<y_2<0)$,

(5) $z_j(1)=1$ $(j=1,2)$ and

(6) except for $z_j(0)$ and $z_j(1)$ we have $C\subset\{\mathrm{Re}\,z>0\}\cap\{\mathrm{Im}\,z>0\}\cap D$ and $C\subset\{\mathrm{Re}\,z>0\}\cap\{\mathrm{Re}\,z>0\}\cap D$. Further, take any two sequences of real numbers $\{a^2_n\}$ and $\{K_n\}$ such that

(7) $a^2_n>a_{n-1}a_{n+1}$ for all n and $0<a_n$ M and

(8) $K_n \geq 1$ and $\lim K_n = +\infty$.

Then, there exists a regular function f, satisfying

(9) $|f(z)| \geq K_n$ on the circle $|z| = a_n$

and

(10) $Z(f) \subset R$

where R denotes the bounded domain determined by C_1, C_2 and the line segment

$$\{z = x + iy \,;\, x = 0,\ y_2 \leq y \leq y_1\}.$$

PROOF. We put $(a_{n+1} - a_n)/(a_{n+1} + a_n) = b_n$ and then clearly $1 > b_n \downarrow 0$. Now by virtue of (4), (5) and (6), we can choose c_n (n=1,2,3,,,,) so small that the domain R includes two line segment

$$\{z = r\exp(ic_n) \,;\, 0 \leq r \leq a_{n+2}\}$$

and

$$\{z = r\exp(-ic_n) \,;\, 0 \leq r \leq a_{n+2}\}$$

and further a circular arc

$$\{z = a_{n+2}\exp(ic) \,;\, -c_n < c < c_n\},\ \text{say } s(a,\ c).$$

Needless to say, we may assume that

$$0 < c_{n+1} < c_n < / \pi / 2 \text{ and } \tan c_n / 2 < b_{n+1}.$$

Now consider the annular sector $D_n = D(a'_{n-1},\ a_{n+1} \,;\, c_{n-1})$

where $a'_{n-1} = a^2_n / a_{n+1}$ and $D(a,\ b \,;\, c') = \{z = r\exp(c) \,;\, a < r < b,\ -c' < c < c'\}$.

Moreover set $s(a,\ c_1,\ c_2) = \{z = a\exp(c) \,;\, c_1 < c < c_2\}$, $s(a_n,\ c_n) = s_n$ and $s(a_n,\ c_n,\ c_{n-1}) \cup s(a_n,\ -c_{n-1},\ -c_n) = S_n$. And further consider

$$d_n = \{|z| = a_n\} - s_n.$$

Then making a slight modification of a standard technique of Bagemihl-Seidel based on the Mergelyan approximation theorem, we can construct a regular function g such that

(15) $g(z) \neq 0$ in $D: |z| < 1$ and $|g(z)| > 2Kn$ on d_n.

Next we choose e_n with $e_n > 0$ and $\sum_n e_n = e_0 < 1/4$. Then we can find a rational function $p_1(z)$, with its only pole in the segment (a_1, a_2) such that

(12) $|p_1(z)| > 2/r^2_1$ on s_1 where $r_1 = \min\{1/2K_1, \min|g(z)| \text{ for } z \in s_1\}$,

(13) $\mathrm{Re}\, p_1(z) > 0$ on S_1

and

(14) $|p_1(z)| < e_1$ on $\{|z| < +\infty\} - D_1$.

Our desire is to approximate $p_1(z)$ by a regular function in $D - D_1$ minus a certain narrow domain including the segment $[a_2, 1)$ pointed at $z = 1$. Since $p_1(z)$ has its only pole in the segment (a_1, a_2), we can sweep out the poles to the boundary point $z = 1$ and consequently obtain a function $h_1(z)$, regular in D, such that

(12)' $|h_1(z)| > 1/r^2_1$ on s_1,

(13)' $\mathrm{Re}\, h_1(z) > -e_1$ on S_1

and

(14)' $|h_1(z)| < 2e_1$ on $D - D_1 - \cup D(a_k, a_{k+1}; c_{k+1}) - \cup s'_k$

where s'_k denotes the arc $\{z = a_k \exp(ic); -c_{k+1} < c < c_{k+1}\}$. Now we shall inductively construct rational functions $p_n(z)$ and $h_n(z)$ as follows. Let $t_n = \sum_k \max\{|h_k(z)|; z \in \bigcup_{k=1}^{n} s_k\}$

Then we get a rational function $p_n(z)$, with its only pole in the open segment (a_n, a_{n+1}), such that

(15) $|p_n(z)| > 2/r^2_n + 2t_n$ on s_n where $r_n = \min\{1/2K_n, \min|g(z)| \text{ for } z \text{ in } s_n\}$

(16) $\mathrm{Re}\, p_n(z) > 0$ on S_n and

(17) $|p_n(z)| < e_n$ on $\{|z| < +\infty\} - D_n$

Then we can find a regular function $h_n(z)$ such that

(15)' $|h_n(z)| > 1/2r^2_n + t_n$ on s_n,

(16)' $\mathrm{Re}\, h_n(z) > -e_n$ on S_n and

(17)' $|h_n(z)| < 2e_n$ on $D - D_n - \cup D(a_k, a_{k+1}; c_{k+1}) - \bigcup_{k=n+1}^{\infty} s'_k$

By virtue of (17)' the series $\sum_n h_n(z)$ uniformly converges on any compact subset of D and hence we obtain a function $h(z) = 1 + \sum_n h_n(z)$, regular in D. Now consider

$$f(z) = g(z)h(z).$$

Then we can find that

$$|f(z)| > 1/r_n - 2r_n \text{ on } s_n \text{ and } |f(z)| > 1/2\,|g(z)| \text{ on } d_n.$$

Consequently, from (11) and the definition of r_n stated in (15), we get that

$$|f(z)| > K_n \text{ on } |z| = a_n.$$

As for the distribution of zeros $f(z)$, remember that $g(z) \neq 0$ in D and note that $\bigcup_n D_n \subset R$. Further, by virtue of (17)', we have

$$|h(z)| > 1/2 \text{ in } D - \bigcup_n D_n.$$

Thus we see that $f(z)$ does not vanish outside of R. This completes the proof.

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