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The Characterization of Zeros of A Strongly Annular Function on Hyperbolic Riemann Surfaces

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1 Introduction

Let W be an open Riemann surface and W_n a regular exaustion for W. Namely, each W_n is supposed to be a compact regular region of W, in other words, a region whose boundary dW_n consists of a finite number of closed analytic Jordan curves and the sequence W_n must satisfy two conditions such that $\overline{W}_n \subset W_{n+1}$ and $\bigcup W_n = W$. For a holomorphic function f defined on W, f will be called an annular function if and only if there exists a regular exaustion W_n such that the minimum modulus of $f(z)$ on dW_n increases to the infinity as n increases to the infinity. In this situation, our concern is restricted to the characterization of zeros of an annular function f. The reason is that the strength or the weakness of the boundary dW seems to restrict or loosen the distribution of zeros of an annular function respectively.

For example, when W is a complex plane, which is a parabolic Riemann surface, the well-known Wieman theorem shows that if an entire function, which is not constant, has an order $a(f)$ less than $1/2$, then there exists a sequence r_n such that $m(r_n) \to \infty$. Here $m(r) = \min\{|f(z)|; |z| = r\}$,

$$a(f) = \overline{\lim_{r\to\infty}}\{\log\log M(r)\}/\log r$$

and $M(r) = \max\{|f(z)|; |z| = r\}$. Therefore, if we take an arbitrary sequence z_n such that the covergent index of $|z_n|$, which is equal to the order of f, satisfies the following inequality

$$a(f) = \inf\{k; \Sigma|z_n|^{-k} < \infty\} < 1/2.$$

Furthermore, we define an entire function by the infinite product $\Pi(1 - z/z_n)$. Then we find that $m(r_n) \to \infty$, where $r = |z_n|$. Since the above condition is independent of z_n, we can conclude that the plane has the week boundary in the following sense : the distribution of the zeros of some kinds of entire functions is not restricted in spite of r_n such that $m(r_n) \to \infty$.

When there exists a Green's function $g(z)$ on a hyperbolic Riemann surface W and we can choose a regular exaustion W_n such that $W_n = \{z; g(z) > e_n\}$ where $e_n \downarrow 0$ if $n \to \infty$, we can define a strongly annular function, namely, we say that f will be called strongly annular if and only if the minimum modulus of $f(z)$ on dW_n for the region W_n defined as above increases to the infinity if $n \to \infty$. Speaking with respect to this definition, in the case of hyperbolic Riemann surfaces, the Wieman theorem gives us a positive answer in the sense that given arbitrarily a sequence z_n of complex numbers, satisfying some condition, we can find a strongly annular function $f(z)$ such that

$$Z(f) = \{z; f(z) = 0\} = \{z_n\}$$

Hence this is the reason why we are interested in the case where W is a hyperbolic open Riemann surface.

2 An open problem

As it is seen from the situation stated in the introduction, the characterization of zeros of annular functions seems to be deeply complicated and very difficult. The main reason may be that the fact a function must be holomorphic is surely qualitative. Therefore, the most general sufficient condition that a sequence z_n coincides with the zeros of an annular function f seems to be far away. Consequently we restrict ourselves to the case where W is the unit disk : $|z| < 1$ on the complex plane. Thus our problem is given as follows: For a given sequence z_n in the unit disk D: $|z| < 1$, find a sufficient condition so that

$$Z(f) = \{z_n\}$$

3 Some results

With respect to the above problem, Bonar-Carroll [1] proved the following results which seems to give a partial answer.

THEOREM. Let $f(z)$ be holomorphic in $D : |z| < 1$. If the zero set $Z(f)$ of f is distributed equi-radially, it is impossible that f becomes a strongly annular function. such that $Z(f) \subset [0,1]$.

In spite of this remark, however, Barth,Bonar-Carroll[2] and Osada[3] proved independently the following interesting

THEOREM. Let R be an arbitrary Jordan region including [0,1], in $D : |z| < 1$. Then there exists a strongly annular function f such that

$$Z(f) \subset R.$$

The difference between the above remark and this theorem seems to be almost nothing. But, to connect the two observations, we have only to show

THEOREM. Let f be holomorphic in $D : |z| < 1$ and suppose that $Z(f) \subset [0,1)$. Then, for an arbitrary Jordan curve J enclosing the origin $z = 0$ and symmetric with respect to the real axis , we have

$$\min\{|f(z)|; z \in J\} < \exp\{\Re a_0 + |a_1|\}$$

where a_0 and a_1 are real numbers such that

$$f(z) = z^p \exp\{a_0 + a_1 z + \cdots\cdots\}.$$

PROOF. We give only the outline of the proof. At first, consider a holomorphic function $G(z)$ such that

$$f(z) = z^p G(z) \exp\{a_0 + a_1 z\}.$$

Clearly $G(0) = 1$. Then, we get

$$\log G(z) = a_1 z + a_2 z^2 + \cdots\cdots .$$

Here, we denote by $z_1, \cdots\cdots, z_n$ the zeros of G in the interior D of J. Then, we find

$$\min\{\log|G(z)|; z \in J\} < -\Sigma \log|h(z_n)| - 1/2|\Sigma\{h(z_n)^{-1} - \overline{h(z_n)}\}|$$

where we denote by $-\log|h(z)|$ the Green function for D_J with pole at $z = 0$. Since J is symmetric with respect to the real axis and each z_n is positive, $h(z_n)$ is also positive if we let $h'(0) > 0$. Hence, we get the required inequality.

4 The principal result

Now we prove the following

THEOREM. Let f be a holomorphic function in $D; |z| < 1$ and suppose that its Maclaurin coefficients are all real. If we have

$$Z(f) \subset [0,1),$$

then, $f(z)$ can not become a strongly annular function.

PROOF. Let J be any closed Jordan curve enclosing the origin $z = 0$ and consider the open set $D - (-1,1)$, whose boundary contains the origin and which is included in the upper half of D. Then, its boundary contains a Jordan arc J_0 which is a subarc of J and which connects a point of (0,1) with one of (-1,0) in the half disk. Hence we denote by $\overline{J_0}$ the reflection of J_0 with respect to the diameter (-1,1) and consider $J_0 \cup \overline{J_0}$.Thus our assertion follows.

As we have proved, there does not exist a strongly annular function such that

$$Z(f) \subset [0,1)$$

Therefore, it is so natural to raise the following open problem: Does there exist an annular function such that $Z(f) \subset [0,1)$?

References

[1] D.D.Bonar and F.W.Carroll, Distribution of a-points for unbounded analytic functions, J. reine angew. Math. 253(1972), 141-145.

[2] K.Barth, D.D.Bonar and F.W.Carroll, Zeros of strongly annular functions, Math.Z. 144(1975), 175-179.

[3] A.Osada, On the distribution of a-points of a strongly annular function, Pacific J. Math. 63(1977).