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# SOME BANACH SPACE ENDOWED A NORM INDUCED BY INFINITE SERIES WITH TERMS OF ABSOLUTE VALUES OF HOLOMORPHIC FUNCTIONS AT A GIVEN SEQUENCE IN A DOMAIN OF AN OPEN RIEMANN SURFACE

AKIO OSADA

**1.** Let $X$ be a complex vector space and denote by $\|\cdot\|$ a norm on $X$, namely, a non negative function defined on $X$ such that

(1) $\| x \| = 0$ if and only if $x = 0$,
(2) $\| x + y \| \leq \|x\| + \|y\|$,
(3) $\| \lambda x \| = |\lambda| \, \| x \|$.

If this norm $\|\cdot\|$ satisfies a condition such that for any sequence $\{x_n\}$ having $\| x_n - x_m \| \to 0$ $(n, m \to 0)$, there exists an element $x$ in $X$ satisfying

$\| x_n - x \| \to 0$ $(n \to \infty)$, we call $X$ a Banach space endowed$\|\cdot\|$.

Now, let $D$ be a domain on an open Riemann surface and denote by $H(D)$ the family of all holomorphic functions on the surface. Moreover, we take a sequence $\{z_n\}$ on the surface and consider the set of all holomorphic functions in $H(D)$ such that

$$\sum_{n=1}^{\infty} |f(z_n)| < +\infty.$$

We denote by $\| f \|$ the sum of this infinite series. The aim of this article is to characterize the sequence $\{z_n\}$ such that $C(D)$, i.e., the set defined above becomes a Banach space.

First of all, we consider the case where the sequence $\{z_n\}$ clusters at no points in $D$.

**2.** Before entering the analysis, we recall the uniqueness theorem; Let $f$ be holomorphic in $D$ and $\{z_n\}$ a sequence converging a point $c$. If

$f(z_n) = 0$ for any $n$, $f(z)$ vanishes at any point $z$ in $D$. To see this, we first expand $f$ at $z = c$ :

$$f(z) = f(c) + a_1(z-c) + a_2(z-c)^2 + \cdots$$

Since $f$ is continuous,

$$f(c) = \lim_{n\to\infty} f(z_n) = 0.$$

Next, we consider a holomorphic function

(1) $$f_1(z) = a_1 + a_2(z-c) + \cdots$$

Then, obviously

$$f_1(z) = f(z)/(z-c) \ (z \neq c)$$

Therefore, $f_1(z_n) = 0$ for any $n$. Since $f(z)$ is also continuous

$$a_1 = f_1(c) = \lim_{n\to\infty} f_1(z_n) = 0.$$

On the other hand, noting (1) and setting as before

$$f_2(z) = a_2 + a_3(z-c) + \cdots$$

we find $f_2(z) = f_1(z)/(z-c)$ and therefore $f_2(z_n) = 0$ for any $n$. Hence,

$$a_2 = f_2(c) = \lim_{n\to\infty} f_2(z_n) = 0.$$

After all, we can conclude that for any $n, a_n = 0$ and that $f(z)$ vanishes in a neighborhood of $z = c$. To show that $f(z) = 0$ for any point $z$, we first connect $z$ to $c$ using an analytic Jordan curve and then we can derive the above property making use of the standard methods.

Now we back to our consideration and try to find the condition such that the space $C(D)$ becomes a Banach space with respect to $\| f \|$. At first, we treat the condition (1). Suppose that $\| f \| = 0$ and then we get $f(z_n) = 0$ for any $n$. Using the uniqueness theorem and recalling that the sequence $\{z_n\}$ converges a point $c$, we can conclude that $f$ vanishes at any point of $D$.

Next, we consider the validity of the condition (2), in other words, we show

$$\| f+g \| = \sum_{n=1}^{\infty} |f(z_n) + g(z_n)| \leq \sum_{n=1}^{\infty} |f(z_n)| + \sum_{n=1}^{\infty} |g(z_n)| = \| f \| + \| g \|$$

which is an easy consequence of an elementary inequality. Finally, we have to prove the most important condition (3), namely, the fact that for any sequence $\{f_n\}$ such that

(2) $$\| f_n - f_m \| \to 0 \ (n, m \ \to \infty)$$

there exists a holomorphic function $f$ in $H(D)$ satisfying

(3) $\| f_n - f \| \to 0 \ (n \to \infty)$.

First of all, recalling

$$\| f_n - f_m \| = \sum_{i=1}^{\infty} |f_n(z_i) - f_m(z_i)| \to 0 \ (n, m \to \infty)$$

we note that for any $j$ fixed,

$$|f_n(z_j) - f_m(z_j)| \to 0 \ (n, m \to \infty)$$

Then, there exists a number $w_j$ such that

$$f_n(z_j) \to w_j \ (n \to \infty).$$

To see this, we have only to show that for any sequence $\{a_n\}$ of real numbers such that

$$|a_n - a_m| \to 0 \ (n, m \to \infty)$$

there exists a number a such that $a_n \to a \ (n \to \infty)$.

This is ,however, the fundamental property of the set of the real numbers, namely, the completion of the set.

Next, we have to prove the existence of a holomorphic function f such that

$$f(z_j) = w_j.$$

For this aim, we first recall the famous Mittag-Leffler theorem and then apply the Weierstrass theorem. Namely, we construct holomorphic functions $g_n(z)$ such that

(4) $g_n(z_j) = 0 \ (j \neq n)$

(5) $g_n(z_n) \neq 0$

Furthermore, we choose $h_n(z)$ such that

(6) $h_n(z)$ is almost equal to zero in $|z| < r_n$,

(7) $\Re h_n(z) > k_n >$ in a neighborhood of $z_n$.

Here, positive constants $r_n$ or $k_n$ are taken sufficiently or arbitrarily large respectively. Now that we have constructed $g_n(z)$ and $h_n(z)$ as above, we have only to set

$$f(z) = \sum_{n=1}^{\infty} e_n g_n(z) \exp\{h_n(z)\}$$

where $e_n$ are suitably chosen positive constants. This function $f(z)$ should be the requested one and satisfy the conditions (3) and (2) which is the most important part of this article. In other words, we have only to show that for any positive number $\epsilon$, there exists a natural number $n_0$, such that

(8) $$\sum_{j=1}^{\infty} |f_n(z_j) - f(z_j)| < \epsilon$$

for any $n > n_0$. First of all, we note that for any $\rho > 0$, there is a natural number $n_1$ such that

(9) $$\sum_{j=1}^{\infty} |f_n(z_j) - f_m(z_j)| < \rho \ (n, m > n_1)$$

which is a direct consequence of the condition (2). By virtue of (5), for any $j$, we see that

$$\sum_{i=1}^{j} |f_n(z_i) - f_m(z_i)| < \rho.$$

Here, letting $m \to \infty$, we get

$$\sum_{i=1}^{j} |f_n(z_i) - f(z_i)| < \rho.$$

Since $j$ is taken arbitrarily, we obtain

$$\sum_{i=1}^{\infty} |f_n(z_i) - f(z_i)| < \rho.$$

Moreover, since $\rho$ is given arbitrarily, we see

$$||f_n - f|| \to 0 \ (n \to \infty)$$

which is the desired conclusion. Thus we get the following

**THEOREM.** The space $C(D)$ is a Banach space with respect to the norm $||f||$.

**3.** When the sequence $\{z_n\}$ clusters at a point of the boundary of $D$, the uniqueness theorem does not hold necessarily. Therefore, it ma be very difficult to get the same result obtained above. We may restrict our consideration to the family of all $f$ such that if $f$ vanishes at any point $z$, $f$ must vanish at every point of $D$.

# References

[1] K. Hoffman, Banach spaces of analytic functions, Prentice Hall.