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SOME REMARK ON BIJETIVE MAPS ASSOCIATED WITH THE HARMONIC DIMENSION CONSIDERED BY HEINS OF THE KEREKYART BOUNDARY OF AN OPEN RIEMANN SURFACE

AKIO OSADA

1. We consider the Kerekjart ideal boundary β of an open Riemann surface R to extend the definition by Heins. For a subset γ of β we make an essential assumption that γ has zero capacity and is isolated from $\delta = \beta - \gamma$. We denote by $P(\Omega)$ $(UV\Omega_i)U(UW\Omega_j)$ the partition of the components of $R - \bar{\Omega}$, induced by $P = \gamma + \Sigma\delta_j$ where δ_j is a component of δ. Here $V\Omega_i$ are components whose derivations are contained in γ and $W\Omega_j$ are the remaining parts.

We consider the normal operator L_1 with respect to $R - \bar{\Omega}$ associated with the partition P. Let q be a function which is continuous on $R - \bar{\Omega}$ and harmonic in $R - \bar{\Omega}$. Then q will be called of L_1-type in $W_\Omega = UW_{\Omega_j}$ if $q = L_1 q$ in W_Ω. It is easy to see that this property is independent of the choice of Ω. To show this , denote by L_1,$L_1^{'}$ normal operators with respect to $R - \bar{\Omega}$, $R - \bar{\Omega}'$ respectively. Without loss of generality ,it may be assumed that $\bar{\Omega} \subset \Omega^{'}$. At first suppose that q is of L-type in W_Ω. We have only to prove that $q = L_1^{'} q$ in $W_{\Omega_1^{'}}$. Next we orient $\gamma_\Omega = \partial V_\Omega$, $\delta_{\Omega_j} = \partial W_{\Omega_j}$ positively with respect to Ω. Here we take two regular exhaustions W_n, $W_n^{'}$ of $W_{\Omega_j}W_{\Omega_j^{'}}$ whose boundary consists of $\delta_n - \delta_{\Omega_j}$, $\delta_n - \delta_{\Omega_j^{'}}$ respectively. Furthermore we consider normal operators L_{1n}, $L_{1n}^{'}$ with respect to W_n, $W_n^{'}$ respectively. Then by Green's formula

$$||L_{1n}q - L_{1n}^{'}q||^2_{W_n^{'}} = -\int_{\delta_{\Omega_j^{'}}} (L_{1n}q - q) * d(L_{1n}q - L_{1n}^{'}q)$$

Since q is of L_1-type in W_Ω, $L_{1n}q$ converges uniformly to q on $\delta_{\Omega_j^{'}}$. Hence

$$L_1^{'} = q \text{ in } W_{\Omega_j^{'}}.$$

Conversely let $q = L_1^{'} q$ in $W_{\Omega'}$. We set $q_n^{'} = L_{1n}^{'} q$ in $W_n^{'}$ and $q_n^{'} = q$ in $W_n - W_n^{'}$. By another application of Green's formula,

$$\|L_{1n}q - q_n'\|^2_{W_n} = \|L_{1n}q - q\|^2_{W_n - W_n'} + \|L_{1n}q - L_{1n}'q\|^2_{W_n}$$
$$= \int_{\delta_{\Omega_j'}} (L_{1n}q - q) * d(L_{1n}q - L_{1n}'q)$$

Since $L_{1n}'q - q$ has its harmonic extension across $\delta_{\Omega_j'}$ and $L_1'q = q$ in $W_{\Omega'}$,

$$\|L_{1n}q - q\|^2_{W_n} \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.$$

Thus we obtain that

$$L_1 q = q \quad \text{in } W_{\Omega_j}.$$

2. Let $HP_0(V_\Omega)$ be the totality of non-negative harmonic functions u in $V_\Omega = UV_{\Omega_i}$ which vanish continuously on $\gamma_\Omega = \partial V_\Omega$. Then we may extend u to be identically zero in W_Ω. Moreover we consider the following two families of functions. The first family N_Ω consists of all $u \in HP_0(V_\Omega)$ such that

$$\int_{\gamma_\Omega} *du = 2\pi$$

The second one is the family F of $q \in H(R - \zeta)$ having the following properties:
(1) $q|D = \log|z - \zeta| + h(z)$ with $h \in H(D)$ and $h(\zeta) = 0$
(2) q is of L_1-type
(3) q is bounded from below near ζ.
In addition to the obvious fact that N_Ω and F are convex, they are related to each other as follows.

LEMMA There exists a unique bijective map T of N_Ω onto F satisfying
(4) $T(\lambda u + (1 - \lambda)v) = \lambda Tu + (1 - \lambda)Tv$ for $u, v \in N_\Omega (0 < \lambda < 1)$
(4) $Tu - u$ is bounded in V_Ω.

PROOF. To see the uniqueness, let T_1 and T_2 be such maps and set $p = T_1 u - T_2 u$ for $u \in N_\Omega$. Then p is regular at ζ and of L_1-type in W_Ω.

By (5) p is bounded in V_Ω for $p = (T_1u - u) - (T_2u - u)$. Hence so is $p - L_1p$. Since cap $\gamma = 0$ and $p = L_1p$ on γ_Ω, we see that $p = L_1p$ in V_Ω. Thus p reduces to a constant.
By (1), p is identically zero. Now let L be the direct sum of the normal operator L and the Dirichlet operator with respect to the parametric disc D (Sario). For $u \in N_\Omega$ we take the singularity function $s(u)$ on $(R - \bar{\Omega})U(D - \zeta)$ defined by

$$s(u) = u \quad \text{in } R - \bar{\Omega}$$

and

$$s(u) = \log|z - \zeta| \quad \text{in } D - \zeta.$$

Since the total flux of $s(u)$ is zero , the equation

$$p - s(u) = L(p - s(u))$$

has a solution $p(u)$ on R, up to an additive constant . Then we normalize $p(u)$ so as to satisfy the condition (1) and set $T(u) = p(u)$. It is easy to see that $Tu \in F$. The property in (1) and (2) follows easily from the definition of T. Next, to see the injectivity , suppose that $Tu = Tv$. Then $u - v = (u - Tu) - (v - Tv)$ is bounded in V_Ω. Since $u = v$ on γ_Ω and cap $\gamma = 0$, we see that $u = v$ in V_Ω. Finally in order to show the surjectivity, let $q \in F$. We denote by Bq the bounded harmonic function in V_Ω with the boundary values $q|\gamma_\Omega$. Because of zero capacity , there exists the only such one. Set $u = q - Bq$ in $V\Omega$ and $u = 0$ in $W\Omega$. However, since q is of L_1-type and bounded from below near γ, we conclude that $u \in N_\Omega$. It remains to prove that

$$q - s(u) = L(q - s(u)) \text{ in } (R - \bar{\Omega})U(D - \zeta).$$

By the definition of u, we have that in $V_\Omega q - s(u) = Bq$ and $L_1(q - s(u)) = L_1(q)$. Moreover $Bq - L_1(q)$ is bounded in V_Ω and vanishes on γ_Ω. Therefore we see that

$$Bq = L_1q.$$

On the other hand, $q - s(u) = q$ and $L_1(q - s(u)) = Lq$. Furthermore, by (2) we get

$$q = L_1q.$$

Accordingly it is obvious that the same equality holds in $D - \zeta$. Thus the proof is complete.

3. As an immediate consequence of the lemma, we obtain the next theorem ,which tells us the following observation ; Since the convex structure of N_Ω is preserved by L, the number of minimal functions in N_Ω is independent of the choice of Ω. By extending the Heins definition we call it the harmonic dimension of γ, which we shall denote by d_γ.

THEOREM. Let Ω_1 and Ω_2 be regular regions.
Then there exists a unique bijective map L of N_{Ω_1} onto N_{Ω_2} satisfying
(6) $L(\lambda u_1 + (1-\lambda)u_2) = \lambda L u_1 + (1-\lambda) L u_2$ for $u_1, u_2 \in N_{\Omega_1}$ and $0 < \lambda < 1$
(7) $Lu_1 - u_1$ is bounded near γ.

References

[1] M.Heins, A lemma on positive harmonic functions, Ann. of Math. vol. 52 (1950), 568-573.

[2] L. Sario, A linear operator method on arbitrary Riemann surfaces, Trans. Amer. Math. Soc. vol. 72 (1952), 281-295.